Limit of sequences

1. Prove the following :

(a) $\lim\limits_{n\to\infty}\dfrac{10000n}{n^2+1}=0$ **(b)** $\lim\limits_{n\to\infty}\left(\sqrt{n+1}-\sqrt{n}\right)=0$ **(c)** $\lim\limits_{n\to\infty}\dfrac{\sqrt[3]{n^2}\sin(n!)}{n+1}=0$

(d) $\lim\limits_{n\to\infty}\dfrac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}=\dfrac{1}{3}$ **(e)** $\lim\limits_{n\to\infty}\dfrac{1+a+a^2+\dots+a^n}{1+b+b^2+\dots+b^n}=\dfrac{1-b}{1-a}$ ($|a|<1$, $|b|<1$)

2. Prove the following :

(a) $\lim\limits_{n\to\infty}\left(\dfrac{1}{n^2}+\dfrac{2}{n^2}+\dots+\dfrac{n-1}{n^2}\right)=\dfrac{1}{2}$ **(b)** $\lim\limits_{n\to\infty}\left|\dfrac{1}{n}-\dfrac{2}{n}+\dfrac{3}{n}-\dots+(-1)^n\dfrac{n}{n}\right|$ does not exist

(c) $\lim\limits_{n\to\infty}\left[\dfrac{1^2}{n^3}+\dfrac{2^2}{n^3}+\dots+\dfrac{(n-1)^2}{n^3}\right]=\dfrac{1}{3}$ **(d)** $\lim\limits_{n\to\infty}\left[\dfrac{1^2}{n^3}+\dfrac{3^2}{n^3}+\dots+\dfrac{(2n-1)^2}{n^3}\right]=\dfrac{4}{3}$

(e) $\lim\limits_{n\to\infty}\left[\dfrac{1}{2}+\dfrac{3}{2^2}+\dfrac{5}{2^3}+\dots+\dfrac{2n-1}{2^n}\right]=3$ **(f)** $\lim\limits_{n\to\infty}\left[\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dots+\dfrac{1}{n(n+1)}\right]=1$

(g) $\lim\limits_{n\to\infty}\left(\sqrt{2}\sqrt[4]{2}\sqrt[8]{2}\cdots\sqrt[2^n]{2}\right)=2$

3. Prove the following :

(a) $\lim\limits_{n\to\infty}\dfrac{n}{2^n}=0$ **(b)** $\lim\limits_{n\to\infty}\dfrac{2^n}{n!}=0$ **(c)** $\lim\limits_{n\to\infty}\dfrac{n^k}{a^n}=0$ ($a>1$, k = constant)

(d) $\lim\limits_{n\to\infty}\dfrac{a^n}{n!}=0$ **(e)** $\lim\limits_{n\to\infty}nq^n=0$ **(f)** $\lim\limits_{n\to\infty}\sqrt[n]{a}=1$ ($a>0$)

(g) $\lim\limits_{n\to\infty}\sqrt[n]{n}=1$ **(h)** $\lim\limits_{n\to\infty}\dfrac{1}{\sqrt[n]{n!}}=1$

4. If $\lim\limits_{n\to\infty}a_n=a$, prove that $\lim\limits_{n\to\infty}|a_n|=|a|$. Is the converse true ?

5. If $\lim\limits_{n\to\infty}a_n=a$ and $\Sigma_n=\dfrac{a_1+a_2+\dots+a_n}{n}$, prove that $\lim\limits_{n\to\infty}\Sigma_n=a$.

6. If $\lim\limits_{n\to\infty}a_n=a$ ($a>0$) , prove that $\lim\limits_{n\to\infty}\log_e a_n=\log_e a$.

7. If $\lim\limits_{n\to\infty}a_n=a$ and $a_n>0$ $\forall$ a , prove that $\lim\limits_{n\to\infty}\sqrt[n]{a_1a_2\dots a_n}=a$.

8. If $a_n>0$ $\forall$ a , and $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}$ exists, prove that $\lim\limits_{n\to\infty}\sqrt[n]{a_n}=\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n}$.

9. Prove that $\lim_{n\to\infty}\left(\frac{(2n)!}{(n!)^2}\right)^{1/n} = 4$.

10. By comparing $y_n = \left(1+\frac{1}{n}\right)^{n+1}$ with $x_n = \left(1+\frac{1}{n}\right)^{n}$, prove that $\lim_{n\to\infty} y_n = e$.

11. If $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$, prove that $\lim_{n\to\infty}\left(\sum_{r=0}^{n}\frac{1}{r!}\right) = e$.

12. If $a_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n-1}-\log_e n$, $b_n = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}-\log_e n$, prove that $\{a_n\}$ is monotonic increasing, $\{b_n\}$ is monotonic decreasing , and that both converge to the same limit.

13. If $a_n = \frac{1\times 3\times 5\times\ldots\times(2n-1)}{4\times 6\times 8\times\ldots\times(2n+2)}$, prove that $\lim_{n\to\infty} a_n = 0$.

14. If $a_1 > 2$ and $a_{n+1} = \frac{6}{1+a_n}$, prove that $\lim_{n\to\infty} a_n = 2$.

15. If $a_1 > 1$ and $a_{n+1} = \frac{1}{2}\left(a_n + \frac{1}{a_n}\right)$, prove that $\lim_{n\to\infty} a_n = 1$.

16. If $a_1 > \sqrt{6}$ and $a_{n+1} = \sqrt{6+a_n}$, prove that $\lim_{n\to\infty} a_n = 3$.

17. Prove that the sequence $u_n = \frac{2n-3}{n+2}$ is **(i)** a monotonic sequence, **(ii)** a bounded function.

Find its limit.

18. Which of the following sequences converge ? State the limit to which each of the convergent sequences tends.

(a) $S_n = 1 + (-1)^n$ **(b)** $S_n = \frac{1}{n}\left[1+(-1)^n\right]$ **(c)** $S_n = n\left[1+(-1)^n\right]$ **(d)** $S_n = 1+\frac{1}{n}(-1)^n$ **(e)** $S_n = \frac{1}{n}+(-1)^n$ **(f)** $S_n = \frac{1}{n^2}+\frac{1}{n}(-1)^n$

(g) $S_n = n^2 + (-1)^n n$ **(h)** $S_n = n + (-1)^n n^2$.

For each of the sequences that does not converge, state whether the sequence is bounded or unbounded, and for each unbounded sequence state also whether it is true or false that $S_n \to\infty$ as $n \to \infty$.

19. For each of the following, state whether it is true or false. If the statement is true, prove it; if it is false, give a counter example.

(a) If $a_n \to 0$ as $n \to\infty$, then $\Sigma\, a_n$ converges .

(b) If $\Sigma\, a_n$ converges, then $a_n \to 0$ as $n \to\infty$.

(c) If $\Sigma\, |a_n|$ is convergent, then $\Sigma\, a_n$ is convergent.

(d) If $\Sigma\, a_n$ is convergent, then $\Sigma\, |a_n|$ is convergent.

20. If $a_n = \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}}$, prove that $\lim_{n\to\infty} a_n = 1$.

(Hint : Show that $\frac{n}{\sqrt{n^2+n}} < a_n < \frac{n}{\sqrt{n^2+1}}$)

21. Show that the limits of the following sequences exist and find their limits.

(a) $x_1 = 2^{1/2}, \dots, x_n = (2x_{n-1})^{1/2}$

(b) $x_1 = c^{1/2}, \dots, x_n = (c + x_{n-1})^{1/2}$, $c > 0$.

22. A sequence is defined as follows : $u_{n+1} = \frac{6(1+u_n)}{7+u_n}$, $u_1 = c > 0$.

Prove that **(i)** if $c > 2$, the sequence is monotonic decreasing ;

(ii) if $c < 2$, the sequence is monotonic increasing.

If the sequence converges, find its limit.

23. Prove that, if a and b are constants such that $0 < b \le a$, and $u_n = \sqrt[n]{a^n + b^n}$, then $\lim_{n\to\infty} u_n = a$.

24. A sequence of numbers x_n is defined by $x_1 = h$, $x_{n+1} = x_n^2 + c$, where $0 < c < ¼$ and h lies between the roots a and b ($a < b$) of the equation $x^2 - x + c = 0$.

Prove that $a < x_{n+1} < x_n < b$, and determine the limit of x_n .

25. Show that the limit of the following sequences are all zero :

(a) $\frac{n+1}{n^3+1}$ **(b)** $\frac{\sin n}{n}$ **(c)** $\frac{n+(-1)^n}{n^2-1}$ **(d)** $\frac{1}{n} - \frac{1}{2n} + \frac{1}{3n} - \dots + (-1)^{n+1}\frac{1}{n^2}$ **(e)** $(n+1)^{1/2} - n^{1/2}$

26. Find the limits of the following sequences as $n \to \infty$:

(a) $u_n = \begin{cases} \frac{n-1}{n} & \text{,if } n \text{ is even} \\ \frac{n+1}{n} & \text{,if } n \text{ is odd} \end{cases}$

(b) $u_n = \begin{cases} 3 & \text{,if } n = 3k \\ \frac{3n+1}{n} & \text{,if } n = 3k+1 \\ 2 + \frac{1+n}{3 - n^{1/2} + n} & \text{,if } n = 3k+2 \end{cases}$

(c) $\frac{3n^2+n}{2n^2-1}$

(d) $n^{1/2}$

(e) $1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$

(f) $\log_{10} n$

27. Show that if $\lim_{n\to\infty} x_n = 0$ and $x_n \ne 0$ for every n , then $\lim_{n\to\infty} \frac{1}{x_n} = \infty$.

28. If both x_n and y_n are convergent , is it true that $x_n y_n$ should converge ? How about the converse ?

29. Calculate the limits of :

(a) $\frac{1}{n^2}+\frac{1}{(n+1)^2}+...+\frac{1}{(2n)^2}$ **(b)** $\frac{1+\frac{1}{2}+...+\frac{1}{2^n}}{1+\frac{1}{4}+...+\frac{1}{4^n}}$

(c) $(\sin n!)\left[\frac{n-1}{n^2+1}\right]^{18}-\left[\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+...+\frac{1}{(n-1)n}\right]\left[\frac{2n^2+1}{n^2-1}\right]$ **(d)** $\frac{(-2)^n+3^n}{(-2)^{n+1}+3^{n+1}}$

30. Use the binomial theorem to show that the limits of both $\frac{n}{a^n}$ $(a > 1)$ and $\frac{n^5}{2^n}$ are zero .

31. Find the limits of the followings :

(a) $\frac{\sin\frac{5}{n^2}}{\tan\frac{1}{n^2}}$ **(b)** $\frac{\sin\frac{1}{n}}{\frac{1}{n^2}}$ **(c)** $\frac{\ln\left(1+\frac{2}{\sqrt{n}}\right)}{\sqrt{n}}$ **(d)** $\frac{1+2+...+n}{n^2+1}$ **(e)** $\frac{\sqrt{n^2+1}+\sqrt{n}-\sqrt{n^2-1}}{n+1}$

32. Given that $x_1 = 1$, $x_2 = 2$ and $x_n=\sqrt{x_{n-1}x_{n-2}}$ $(n > 2)$, find $\lim_{n\to\infty} x_n$.

33. If $a > b > 0$, $a_1=\frac{1}{2}(a+b)$, $b_1=\frac{2ab}{a+b}$, $a_n=\frac{1}{2}(a_{n-1}+b_{n-1})$, $b_n=\frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}}$.

Prove that the sequences $\{a_n\}$, $\{b_n\}$ have a common limit $\sqrt{ab}$.

34. If $x_1 > 0$, $x_{n+1}=\frac{3(1+x_n)}{3+x_n}$, show that :

(i) $(x_{n+1} - x_n)$ and $(x_n - x_{n-1})$ have the same sign ; (i.e. $\{x_n\}$ is monotonic .)

(ii) $\left|x_{n+1}-\sqrt{3}\right|<k\left|x_n-\sqrt{3}\right|$, where $0 < k < 1$. Hence $\lim_{n\to\infty} x_n=\sqrt{3}$.

35. Given that $u_{n+1}=\frac{1}{2}\left(u_n+\frac{A^2}{u_n}\right)$, where $0 < A \le u_1$. Prove that :

(i) $u_{n+1} \ge A$ and $u_{n+1} \le u_n$;

(ii) $d_{n+1} = d_n^2$, where $d_n=\frac{u_n-A}{u_n+A}$;

(iii) as n tends to infinity, u_n tends to A .

Hence, calculate $\sqrt{11}$ correct to three places of decimals.